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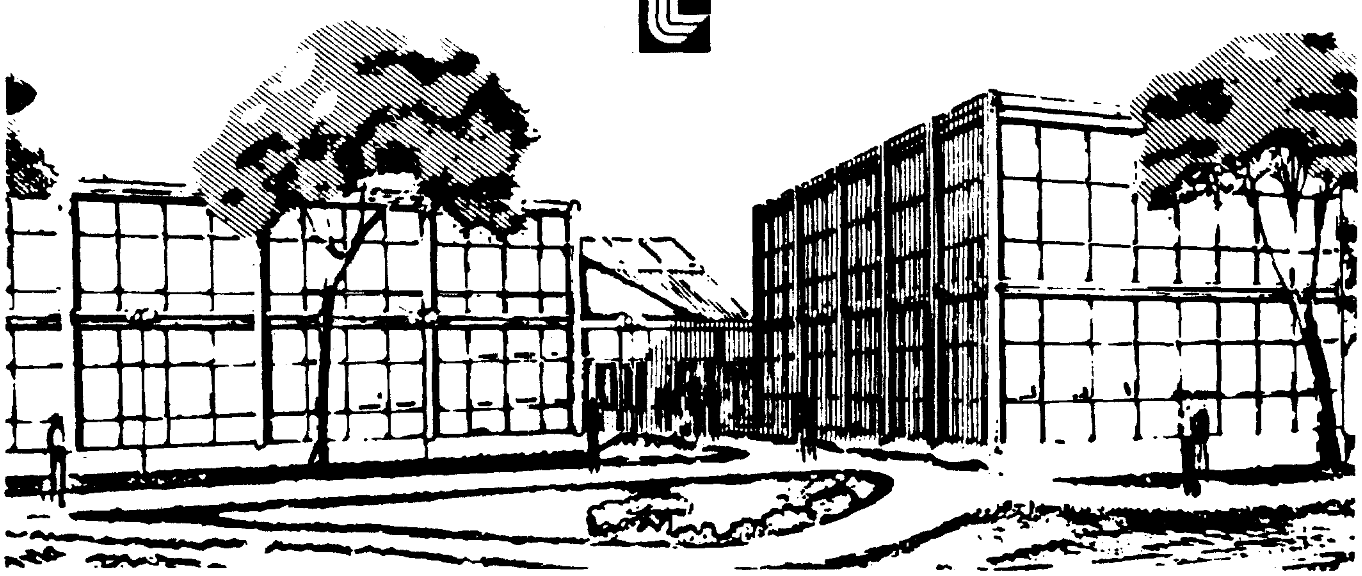

SPECTRAL TRANSFORMATIONS FOR NONSYMMETRIC HALF-PLANE FILTERS*

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I. Introduction

In two dimensions, the problem of producing a transfer function which is stable and which approximates a desired frequency response is usually very intractable. One way around this problem is to begin with a transfer function whose response is easily designed and then transform this transfer function into a new one whose response approximates the given desired response. The spectral transformation procedure [1,2] begins with a transfer function, $F(z_1,z_2)$, whose response we wish to modify, and involves finding two functions, $G(z_1,z_2)$ and $H(z_1,z_2)$ such that the new transfer function $F[G(z_1,z_2),H(z_1,z_2)]$ has the response we desire. As was stated in [1], these two functions are required to be such that they:

1. produce a stable transfer function from a stable transfer function;
2. transform a real rational function into a real rational function;
3. preserve some important characteristics of the frequency response (e.g., ripple magnitudes) while modifying others (e.g., cutoff frequencies).

The functions we will use here are the same as those used in [1]: real rational all-pass functions having the form:

$$G(z_1,z_2) = \frac{z_1^{M_g} z_2^{N_g} C(z_1^{-1},z_2^{-1})}{C(z_1,z_2)} \tag{1}$$

$$H(z_1,z_2) = \frac{z_1^{M_h} z_2^{N_h} D(z_1^{-1},z_2^{-1})}{D(z_1,z_2)}$$

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Such functions obviously satisfy the second condition; to show that they satisfy the third, we first observe that $|G(e^{-i\omega_1},e^{-i\omega_2})| = 1$ and $|H(e^{-i\omega_1},e^{-i\omega_2})| = 1 \ \forall \ (\omega_1,\omega_2) \in [-\pi,\pi]^2$; hence, for each such $(\omega_1,\omega_2) \ \exists \ (u_1,u_2) \in [-\pi,\pi]^2$:

$$G(e^{-i\omega_1},e^{-i\omega_2}) = e^{iu_1} \tag{2}$$

$$H(e^{-i\omega_1},e^{-i\omega_2}) = e^{iu_2}$$

Therefore, the frequency response of the new transfer function, i.e.:

$$F[G(e^{-i\omega_1},e^{-i\omega_2}),\ H(e^{-i\omega_1},e^{-i\omega_2})] \tag{3}$$

is a function which is the composition of the frequency response of the original transfer function, $F(e^{-i\omega_1},e^{-i\omega_2})$, with the following map from $[-\pi,\pi]^2$ to $[-\pi,\pi]^2$:

$$(u_1,u_2) \triangleq [\mathrm{Arg}\{G(e^{-i\omega_1},e^{-i\omega_2})\},\ \mathrm{Arg}\{H(e^{-i\omega_1},e^{-i\omega_2})\}] \tag{4}$$

Since we can view the new frequency response as a composition of the old frequency response with a mapping of the 2-D frequency plane into itself, we conclude that the third requirement is satisfied.

In the quarter-plane case, the conditions for satisfying the first requirement, that the filter be stable, are simple. It was shown in [1] that if the denominator functions of $F(z_1,z_2)$, $G(z_1,z_2)$, and $H(z_1,z_2)$ are all nonzero on $\overline{U}^2 \triangleq \{(z_1,z_2): |z_1| \le 1, |z_2| \le 1\}$, then so will be the denominator function of $F[G(z_1,z_2), H(z_1,z_2)]$. Hence, a sufficient condition for the new transfer function to be stable is that the denominators of the original transfer function and of the functions given by (1) all be nonzero on the unit bidisc. Unfortunately, the conditions for nonsymmetric half-plane filters are not as general, and we must consider special cases.

II. Stability of Spectral Transformations for Nonsymmetric Half-Plane Filters

We will call $A(z_1,z_2)$ a quarter-plane function if it is a 2-D polynominal. On the other hand, we will call $A(z_1,z_2)$ a nonsymmetric half-plane function if it is of the form:

$$A(z_1,z_2) = \sum_{n=0}^{N_a} a(0,n) z_2^n + \sum_{m=1}^{M_a} \sum_{n=-L_a}^{N_a} a(m,n) z_1^m z_2^n \qquad (5)$$

The lower case subscripts on the limits of summation correspond to the upper case letter which is the symbol for the given function, e.g., if the function were $B(z_1,z_2)$, then the limits in (5) would be M_b, N_b, and L_b.

The following lemma is obvious from the definitions given above.

<u>Lemma 1</u>. The sum or product of two quarter-plane functions is a quarter-plane function, the sum or product of two nonsymmetric half-plane functions is a nonsymmetric half-plane function, and the sum or product of a nonsymmetric half-plane function and a quarter-plane function is a nonsymmetric half-plane function.

We will now consider two cases in which spectral transformation preserve stability.

<u>Case I</u>. $F(z_1,z_2) = A(z_1,z_2)/B(z_1,z_2)$ is a quarter-plane transfer function (i.e., $A(z_1,z_2)$ and $B(z_1,z_2)$ are quarter-plane functions) with $B(z_1,z_2)$ satisfying the stability condition $B(z_1,z_2) \neq 0$ on $\bar{U}^2$. The all-pass functions of (1) have denominator functions $C(z_1,z_2)$ and $D(z_1,z_2)$ which are nonsymmetric half-plane functions and which satisfy the stability condition [3] that they be nonzero on $\Theta \triangleq \Delta \cup \Gamma$ where $\Delta \triangleq \{(0,z_2): |z_2| \leq 1\}$ and $\Gamma \triangleq \{(z_1,z_2): |z_1| \leq 1, |z_2| = 1\}$.

In this case, the transfer function $F[G(z_1,z_2), H(z_1,z_2)]$ will be a stable nonsymmetric half-plane transfer function if the positive integers M_g, N_g, M_h, and N_h, which are defined by equation (1), and the non-negative integers M_c, N_c, M_d, and N_d, which are defined by equation (5), satisfy the four inequalities:

$$M_g \geq M_c,\ N_g \geq N_c,\ M_h \geq M_d,\ N_h \geq N_d \qquad (6)$$

To show that the new transfer function is stable, we first clear the fractions from numerator and denominator to give:

$$F[G(z_1,z_2),H(z_1,z_2)] = \frac{A[G(z_1,z_2),H(z_1,z_2)][C(z_1,z_2)]^{\alpha}[D(z_1,z_2)]^{\beta}}{B[G(z_1,z_2),H(z_1,z_2)][C(z_1,z_2)]^{\alpha}[D(z_1,z_2)]^{\beta}} \qquad (7)$$

where $\alpha = \max\{M_a,M_b\}$ and $\beta = \max\{N_a,N_b\}$. The denominator of (7) may be written as:

$$\sum_{m=0}^{M_b} \sum_{n=0}^{N_b} \{b(m,n)[z_1^{M_g} z_2^{N_g} C(z_1^{-1},z_2^{-1})] \cdot [z_1^{M_h} z_2^{N_h} D(z_1^{-1},z_2^{-1})] \cdot [C(z_1,z_2)]^{\alpha-m}[D(z_1,z_2)]^{\beta-n}\} \qquad (8)$$

If the inequalities (6) are satisfied, then $z_1^{M_h} z_2^{N_h} D(z_1^{-1},z_2^{-1})$ and $z_1^{M_h} z_w^{N_h} D(z_1^{-1},z_2^{-1})$ are quarter-plane functions, and since $C(z_1,z_2)$ and $D(z_1,z_2)$ are nonsymmetric half-plane functions, equation (8) is a nonsymmetric half-plane function by lemma 1. The same kind of argument can be applied to the numerator of (7) and so we conclude that (7) has the form of an nonsymmetric half-plane transfer function. To show that (7) is stable, we will show that its denominator is non-zero on Θ. Since we have assumed that $C(z_1,z_2)$ and $D(z_1,z_2)$ are nonzero on Θ, it is only necessary to show that $B[G(z_1,z_2), H(z_1,z_2)]$ is also nonzero on Θ. Since for any fixed z_2 with $|z_2| =1$, $G(z_1,z_2)$ is an analytic function of z_1 on $\{z_1: |z_1| \leq 1\}$ (this follows from the fact that $C(z_1,z_2) \neq 0$ on Γ), and since $|G(z_1,z_2)| = 1 \ \forall \ (z_1,z_2) \in T^2 \triangleq \{(z_1,z_2): |z_1| = |z_2| = 1\}$, it follows from the maximum modulus theorem (for <u>one</u> complex variable) that $|G(z_1,z_2)| \leq 1 \ \forall \ (z_1,z_2) \in \Gamma$. In particular, $|G(0,z_2)| \leq 1$ for

$|z_2| = 1$, and so, using the fact that $G(0,z_2)$ is an analytic function of z_2 on $\{z_2: |z_2| \le 1\}$ and again applying the maximum modulus theorem, we conclude that $|G(0,z_2)| \le 1$ if $|z_2| \le 1$. We next apply exactly the same argument to $H(z_1,z_2)$ to conclude that the moduli of $G(z_1,z_2)$ and $H(z_1,z_2)$ are both ≤ 1 on Θ. Since $B(z_1,z_2) \ne 0$ on $\bar{U}^2$ is assumed, it follows that $B(G(0,z_2), H(0,z_2)] \ne 0$ for $|z_2| \le 1$ and $B[G(z_1,z_2), H(z_1,z_2)] \ne 0$ on Γ. Hence this function is non-zero on $\Theta = \Delta \cup \Gamma$ and the filter is stable.

Case II. $F(z_1,z_2) = A(z_1,z_2)/B(z_1,z_2)$ is a nonsymmetric half-plane transfer function with $B(z_1,z_2) \ne 0$ on Θ. $C(z_1,z_2)$ in equation (1) is a nonsymmetric half-plane function with is non-zero on Θ, and $H(z_1,z_2) = \pm z_2^{N_h}$ where $N_h \ge 1$.

In this case, the transformed filter will have a stable nonsymmetric half-plane transfer function if the inequalities

$$M_g \ge M_c + 1, \; N_g \ge N_c \tag{9}$$

are satisfied. With the fractions cleared, the new transfer function is:

$$F[G(z_1,z_2), z_2^{N_h}] = \frac{A[G(z_1,z_2), \pm z_2^{N_h}][C(z_1,z_2)]^{\alpha}}{B[G(z_1,z_2), \pm z_2^{N_h}][C(z_1,z_2)]^{\alpha}} \tag{10}$$

where $\alpha = \max\{M_a, M_b\}$. The denominator of (10) may be written as:

$$\sum_{n=0}^{N_b} b(0,n)[z_2^{N_h \cdot n}][C(z_1,z_2)]^{\alpha} + \sum_{m=1}^{M_b} \sum_{n=-L_b}^{N_b} \{b(m,n)[z_1^{(M_g-1)} z_2^{N_g} C(z_1^{-1}, z_2^{-1})]^m [C(z_1,z_2)]^{\alpha-m} [z_1^m z_2^{N_h \cdot n}]\} \tag{11}$$

Since the expressions inside each of the square brackets are either quarter-plane or nonsymmetric half-plane functions, by lemma 1 we conclude that (11) is a nonsymmetric half-plane function, and, by applying a similar argument to the numerator, that (10) is a nonsymmetric half-plane transfer function.

To show that (10) is stable, we show that its denominator is non-zero on Θ. $C(z_1,z_2)$ is non-zero on Θ and so it is only necessary to show that $B[G(z_1,z_2), z_2^{N_h}]$ is also non-zero on Θ. Since $M_g \ge M_c + 1$, $G(0,z_2) \equiv 0$, and so $B[G(0,z_2), z_2^{N_h}] = B(0, z_2^{N_h})$. Since $B(0,z_2) \ne 0$ for $|z_2| \le 1$ is assumed, it follows that $B[G(0,z_2), z_2^{N_h}] \ne 0$ for $|z_2| \le 1$ also. Next, using the argument used in case I, we see that $|G(z_1,z_2)| \le 1 \; \forall \; (z_1,z_2) \in \Gamma$. Since $B(z_1,z_2) \ne 0 \; \forall \; (z_1,z_2) \in \Gamma$ is assumed, it follows that $B[G(z_1,z_2), z_2^{N_h}] \ne 0$ on Γ also, hence (10) is stable.

The above two cases are the only ones for which spectral transformations work in general. Case I includes quarter-plane filters as a special case. The inequality constraints required in both cases are necessary to insure that the filter has the form of an asymmetric half-plane filter. The reason the form of $H(z_1,z_2)$ is so limited in case II is that powers of z_2^{-1} occur in $B(z_1,z_2)$; hence, after clearing the fractions in the numerator and denominator of $F[G(z_1,z_2), H(z_1,z_2)]$, there will in general be factors of both $D(z_1,z_2)$ and $z_1^{M_h} z_2^{N_h} D(z_1^{-1}, z_2^{-1})$ in the denominator function. If one of these factors has no zeros on Γ, then in general the other must have a zero on Γ, and so it would be impossible for the denominator to be non-zero on $\Gamma \subset \Theta$.

III. Behavior of Nonsymmetric Half-Plane All-Pass Functions

In section I we showed that the frequency response of a spectrally transformed filter could be described as the composition of the original frequency response function with a mapping from the 2-D frequency plane into itself. This mapping is carried out by the two all-pass functions $G(z_1,z_2)$ and $H(z_1,z_2)$. An important item in the specification of a frequency response function is the number of its pass and stop regions. The number of such regions is determined in large part by the character of the mapping mentioned above. In particular, it is important to know the number of times each all-pass function maps a given curve in the 2-D frequency plane around the unit circle. Pendergrass et al. considered this problem for the quarter-plane case in [1]; here we extend their results to the nonsymmetric half-plane case.

We will consider the all-pass function $G(z_1,z_2)$ of equation (1) with the denominator function $C(z_1,z_2)$ being of the forms given by equation (5). We will also assume that $M_g \geq M_c$ and $N_g \geq N_c$, and that $C(z_1,z_2) \neq 0$ on Θ.

Let $\omega_2 \in [-\pi,\pi]$ be fixed, then $C(z_1, e^{-i\omega_2})$ and $C(z_1, e^{+i\omega_2})$ are both polynomials in z_1 having no zeros on $\{z_1: |z_1| \leq 1\}$. It follows that $z_1^{M_g}(e^{-i\omega_2})^{N_g} \cdot C(z_1^{-1}, e^{i\omega_2})$ is a polynomial in z_1 with all of its M_g zeros in $\{z_1: |z_1| < 1\}$, and that:

$$G(z_1, e^{-i\omega_2}) = \frac{z_1^{M_g}(e^{-i\omega_2})^{N_g} C(z_1^{-1}, e^{i\omega_2})}{C(z_1, e^{-i\omega_2})} \quad (12)$$

has no poles on $\{z_1: |z_1| \leq 1\}$ and all of its M_g zeros in $\{z_1: |z_1| < 1\}$. Applying the Cauchy mapping theorem [4], we conclude that (with ω_2 fixed) as ω_1 increases from $-\pi$ to $+\pi$, the contour produced by $G(e^{-i\omega_1}, e^{-i\omega_2})$ goes around the unit circle M_g times.

Defining

$$N(z_1) = \oint_{|z_2|=1} \frac{\frac{\partial C(z_1,z_2)}{\partial z_2}}{C(z_1,z_2)}\, dz_2$$

We see that since $C(z_1,z_2) \neq 0$ on Γ, $N(z_1)$ is a continuous function on the z_1 unit disc. Furthermore, since (for fixed z_1) $N(z_1)$ counts the number of zeros less the number of poles of $C(z_1,z_2)$ (considered as a function of z_2) on the z_2 unit disc, $N(z_1)$ must be integer valued and therefore constant on the z_1 unit disc. Now, since $C(0,z_2)$ is a polynomial in z_2 and is assumed nonzero on the z_2 unit disc, it follows that $N(z_1) = N(0) = 0$ on the z_1 unit disc. This allows us to conclude that for any fixed z_1 on the z_1 unit disc, $C(z_1,z_2)$, thought of as a function of z_2, has as many zeros as poles on the z_2 unit disc. If we now let $\omega_1 \in [-\pi,\pi]$ be fixed, we see that $C(e^{-i\omega_1}, z_2)$ and $C(e^{i\omega_1}, z_2)$ each have as many zeros as poles on $\{z_2: |z_2| \leq 1\}$. These particular poles and zeros are actually in $\{z_2: |z_2| < 1\}$ because the poles are obviously at the origin and $C(z_1,z_2) \neq 0$ on $T^2 \subset \Gamma$ is assumed. Suppose that (after ω_1 has been fixed and the terms have been collected) the highest positive power of z_2 appearing in $C(e^{+i\omega_1}, z_2)$ is N and the highest negative power is L (clearly $N \leq N_c$, $L \leq L_c$). Then $C(e^{+i\omega_1}, z_2)$ has N + L zeros and L poles, with L poles and L zeros in U and N zeros in the complement of $\{z_2: |z_2| \leq 1\}$ follows that $z_2^N C(e^{+i\omega_1}, z_2^{-1})$ is a polynomial in z_2 with N zeros in $\{z_2: |z_2| < 1\}$ and none on $\{z_2: |z_2| = 1\}$, and therefore that $[e^{-i\omega_1}]^{M_g} z_2^{N_g} C(e^{+i\omega_1}, z_2^{-1})$ is a polynomial in z_2 with N zeros in $\{z_2: |z_2| < 1\}$ and none on $\{z_2: |z_2| = 1\}$. This allows us to conclude that the function:

$$G(e^{-i\omega_1}, z_2) = \frac{[e^{-i\omega_1}]^{M_g} z_2^{N_g} C(e^{i\omega_1}, z_2^{-1})}{C(e^{-i\omega_1}, z_2)} \quad (13)$$

has N_g more zeros than poles in $\{z_2: |z_2| < 1\}$ and no poles or zeros on $\{z_2: |z_2| = 1\}$. By the Cauchy mapping theorem, it follows that (with ω_1 fixed) as ω_2 increases from $-\pi$ to $+\pi$ the contour produced by $G(e^{-i\omega_1}, e^{-i\omega_2})$ goes around the unit circle N_g times.

IV. Discussion

In this paper, we extended some aspects of the theory of spectral transformations to include nonsymmetric half-plane filters. Unfortunately, the results are not as satisfying as those for quarter-plane filters where substituting first quadrant stable all-pass functions into a first-quadrant stable transfer function always yields a new transfer function which is first-quadrant stable. Here spectral transformations where shown to be useful in only two special cases.

Case I is the most important and useful of the two because it is a simple extension of the quarter-plane case: given any first-quadrant stable quarter-plane transfer function, we can substitute arbitrary stable nonsymmetric half-plane all-pass functions, and the resulting nonsymmetric half-plane transfer function will always be stable.

In section III, we showed that the number of times certain lines in the 2-D frequency plane are mapped around the unit circle by $G(e^{-i\omega_1}, e^{-i\omega_2})$ is determined by the integers M_g and N_g. This is exactly the same result developed in [1] for the quarter-plane case. It is interesting to observe that L_c has no effect on the number of times the unit circle is traversed, and so it is possible in the nonsymmetric half-plane case to increase the order of the all-pass function (by adding higher order terms in z_2^{-1} to $C(z_1,z_2)$ without increasing the number of pass and stop bands. Since adding higher order terms in positive powers of z_1 or z_2 will usually increase M_g or N_g (typically $M_g = M_c$ and $N_g = N_c$), we see that the quarter-plane case does not possess this flexibility.

In [1], only the case where ω_1 or ω_2 was fixed at zero and (respectively) ω_2 or ω_1 was allowed to vary was considered, whereas in section III the fixed variable could have any value in $[-\pi,\pi]$. This slight extension of what was done in [1] has an unfortunate consequency for certain applications of spectral transformations. On each vertical line segment stretching from bottom to top of the square $[-\pi,\pi]^2$ the function $G(e^{-i\omega_1}, e^{-i\omega_2})$ must assume the value of each point on the unit circle at least N_g times; similarly, on each horizontal line segment running from one side of the square to the other, the value of each point on the unit circle must be assumed at least M_g times. These restrictions on the behavior of the all-pass function severely limit the types of frequency response which can be produced by a spectral transformation operating on a given transfer function. For example, suppose we are given a stable 1-D lowpass transfer function $H(z)$ (note that $H(z)$ may be interepreted as being the stable quarter-plane transfer function $\hat{H}(z_1,z_2) \triangleq H[z_1]$) upon which we perform a spectral transformation to give the transfer function $H[G(z_1,z_2)]$. From our discussion above we see that on each vertical (horizontal) line in the square $[-\pi,\pi]^2$, the amplitude response $|H[G(e^{-i\omega_1}, e^{-i\omega_2})]|$ must assume both the minimum and maximum values of $H(e^{-i\omega})$ at least $N_g(M_g)$ times. From this we conclude that it is impossible to use this approach to produce a filter whose pass (or stop) band is limited to some small region of the frequency plane (e.g., a lowpass or a highpass filter). It should be noted that the discussion in this paragraph applies to both nonsymmetric half-plane and quarter-plane all-pass functions.

V. References

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